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Moment Equations for a Granular Material in a Thermal Bath

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We compute the moment equations for a granular material under the simplifying assumption of pseudo-Maxwellian particles approximating dissipative hard spheres. We obtain the general moment equations of second and third order and the isotropic moment equations of any order. Our equations describe, in the space homogeneous case, the granular system described by a Boltzmann-like collision term and subject to a Brownian motion due to the interaction with a bath, described by a Fokker–Planck term. The trend to equilibrium is studied in detail.

KEY WORDS: Granular material; Boltzmann equation; thermal bath.

1. INTRODUCTION

In the last few years a notable development of the study of the mechanics of granular materials has occurred, because of their growing importance in the applications (sands, powders, rock and snow avalanches, landslides, grains, fluidized beds). The problems related to the study of fast flows of grain materials, which arise, more and more frequently, in industrial processes and are of growing importance in the study of natural phenomena, have been the object of much attention and have been treated with various methods that differ in rigor and complexity. In the majority of these studies one adopts the assumption of one-dimensional flow and neglects the interaction between grains and air. The various methods applied to these simplified problems have also been used to model other important cases of

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collisional granular motion, including fluidized beds. The methods used in these studies include: (i) development of physical and experimental models; (ii) computer simulations; (iii) kinetic theory.

Actually many recent studies are based on the assumption that, in certain conditions of motion, collisions between particles supply the main mechanism of momentum and energy exchange. This assumption spontaneously suggests an analogy with the kinetic theory of gases. In this theory the particles are of course molecules and there are thus essential differences between the two situations, which must be duly taken into account. In particular, the intermolecular collisions are frequently elastic, whereas this is not a reasonable assumption when dealing with particles of a granular material.

In this paper, we continue the study of the behavior of large systems of inelastic particles on the basis of approximate kinetic equations, proposed in refs. 3 and 4. We consider pseudo-Maxwellian particles approximating dissipative hard spheres, with the aim of studying the equations satisfied by the isotropic moments of the distribution function. Our equations describe, in the space homogeneous case, the system of the aforementioned particles undergoing interparticle inelastic collisions, described by a Boltzmann-like collision term and subject to a Brownian motion due to the interaction with a bath, described by a Fokker–Planck term, as suggested by some recent simulations.[1,6]

We also mention some recent papers[7,8] devoted to mainly numerical studies of pair correlations in a one-dimensional model of inelastically interacting point particles excited by a white noise (diffusion in velocity space). Despite many interesting features[7,8] of one-dimensional models, it is typical in statistical mechanics that these models differ even qualitatively from multidimensional ones. For example, the mean free path of the particle is independent on its diameter in 1d, whereas point particles have no collisions at all in 3d. Our goal in this paper is to study in detail the one-particle distribution function $f(\mathbf{v})$ in velocity space for a more realistic three-dimensional model of inelastic particles in a thermal bath.

We remark that the exact formulas for moments of the collision integral obtained in this paper can be used in many problems. They were already applied (without giving details) by one of the authors (C.C.) to the shear flow of granular materials.[9]

The paper is organized as follows. In Section 2 we recall our basic equations, and perform a Fourier transform of the kinetic equation. In Section 3 we derive exact equations for all tensor moments of order $n \leqslant 3$. Then we concentrate in Section 4 on isotropic solutions and derive exact equations for moments of arbitrarily high order. Then we show that these moments tend to certain values (the moments of a stationary solution) as

$t \to \infty$. Next we derive exact recurrence formulas for the steady state moments and prove that they correspond to a uniquely defined stationary solution of the Fourier transformed equation. We study in Section 5 in more detail the case when the Fokker–Planck term does not include friction. It is proved that the solution constructed in Section 4 is the Fourier transform of a classical nonnegative solution of our kinetic equation (a uniqueness theorem is also established). At the end of Section 5, we discuss some properties of the solution $f(|\mathbf{v}|)$. It is proved that the function $f(|\mathbf{v}| \exp(-\alpha|\mathbf{v}|)) > 0$ is not integrable for some $\alpha > 0$. It is therefore assumed that $|\mathbf{v}|^{-1} \log f(|\mathbf{v}|) \to \alpha$ as $|\mathbf{v}| \to \infty$, and then constructed, at a formal level, the leading asymptotic term of $f(|\mathbf{v}|)$ for large $|\mathbf{v}|$. It is also shown that a small inelasticity expansion[4] can be easily obtained *via* the Fourier transform. The main results of the paper are formulated in Theorems 3.1, 4.1, 4.2, 5.1.

2. THE KINETIC EQUATION AND ITS FOURIER REPRESENTATION

Let $f(\mathbf{v}, t)$ be a distribution function (here $\mathbf{v} \in \mathbb{R}^3$ and $t \in \mathbb{R}_+$ denote the velocity and time variables, respectively) of a spatially homogeneous system of inelastic particles. Following refs. 3 and 4 we describe the system by the pseudo-Maxwellian kinetic equation

$$\frac{\partial f}{\partial t} = B(\rho, t)\, Q(f, f) + L_{FP} f \tag{2.1}$$

where the first term in the right-hand side corresponds to inelastic collisions between particles, whereas the second term is responsible for the interaction of particles with a thermal bath. The explicit form of the first term is given by the following formulas which correct the strong form of the pseudo-Maxwellian collision integral given in ref. 3:

$$Q(f, f) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \int_{S^2} [f(t, \mathbf{v}_*)\, f(t, \mathbf{w}_*)\, J - f(t, \mathbf{v})\, f(t, \mathbf{w})]\, d\mathbf{n}\, d\mathbf{w} \tag{2.2}$$

where $\mathbf{v}_*$, $\mathbf{w}_*$ are the pre-collisional velocities associated to the collision mechanism

$$\begin{aligned} \mathbf{v}_* &= \frac{1}{2}(\mathbf{v}+\mathbf{w}) - \frac{1-e}{4e}(\mathbf{v}-\mathbf{w}) + \frac{1+e}{4e}|\mathbf{v}-\mathbf{w}|\,\mathbf{n} \\ \mathbf{w}_* &= \frac{1}{2}(\mathbf{v}+\mathbf{w}) + \frac{1-e}{4e}(\mathbf{v}-\mathbf{w}) - \frac{1+e}{4e}|\mathbf{v}-\mathbf{w}|\,\mathbf{n}. \end{aligned} \tag{2.3}$$

and J is the Jacobian of the transformation:

$$J = \frac{1}{e^2} \frac{|\mathbf{v}-\mathbf{w}|}{|\mathbf{v}_* - \mathbf{w}_*|} \tag{2.4}$$

Here $0 < e \leqslant 1$ is the restitution coefficient ($e = 1$ for elastic collisions).

Then the weak form of the collision integral coincides with the weak form given in ref. 3:

$$\int d\mathbf{v}\, g(\mathbf{v})\, Q(f, f)$$
$$= \frac{1}{8\pi} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} f(t, \mathbf{v})\, f(t, \mathbf{w}) [g(\mathbf{v}') + g(\mathbf{w}') - g(\mathbf{v}) - g(\mathbf{w})]\, d\mathbf{v}\, d\mathbf{n}\, d\mathbf{w}, \tag{2.5}$$

where $g(\mathbf{v})$ is a test function and $\mathbf{v}'$, $\mathbf{w}'$ are post-collisional velocities given by

$$\begin{aligned} \mathbf{v}' &= \frac{1}{2}(\mathbf{v}+\mathbf{w}) + \frac{1-e}{4}(\mathbf{v}-\mathbf{w}) + \frac{1+e}{4}|\mathbf{v}-\mathbf{w}|\, \mathbf{n} \\ \mathbf{w}' &= \frac{1}{2}(\mathbf{v}+\mathbf{w}) - \frac{1-e}{4}(\mathbf{v}-\mathbf{w}) - \frac{1+e}{4}|\mathbf{v}-\mathbf{w}|\, \mathbf{n}. \end{aligned} \tag{2.6}$$

We denote

$$\rho = \int_{\mathbb{R}^3} f(\mathbf{v}, t)\, d\mathbf{v}, \qquad \rho\mathbf{u} = \int_{\mathbb{R}^3} \mathbf{v} f(\mathbf{v}, t)\, d\mathbf{v}, \qquad 3\rho\theta = \int_{\mathbb{R}^3} |\mathbf{v}-\mathbf{u}|^2 f(\mathbf{v}, t)\, d\mathbf{v} \tag{2.7}$$

where $\rho \in \mathbb{R}_+$, $\mathbf{u} \in \mathbb{R}^3$, and $\theta \in \mathbb{R}_+$ are the density, bulk velocity and temperature of the granular material. Then

$$B(\rho, t) = B(\rho) \sqrt{\theta(t)} \tag{2.8}$$

where $B(\rho)$ is a given positive function of the density ρ (see refs. 3 and 4 for details). This function is irrelevant for our goals since our model preserves the total number of particles ($\rho = \text{const.}$).

Interaction with the bath can be described by one of the following Fokker–Planck terms:

$$L_{FP}^{(1)} f = F \Delta_{\mathbf{v}} f, \qquad L_{FP}^{(2)} f = \frac{1}{\tau} \operatorname{div}_{\mathbf{v}}((\mathbf{v}-\mathbf{u}) f + \theta_b \nabla_{\mathbf{v}} f). \tag{2.9}$$

that were proposed in ref. 4. Thus, all notations in (2.1) are explicitly given in (2.2)–(2.9). Equation (2.1) was first studied in ref. 4 (mainly in the steady case). Our aim is to investigate this equation in more detail.

The main idea is to pass to the Fourier representation of (2.1). Following refs. 2 and 3 we introduce the characteristic function

$$\phi(\mathbf{k}, t) = \int_{\mathbb{R}^3} f(\mathbf{v}, t)\, e^{-i\mathbf{k}\cdot\mathbf{v}}\, d\mathbf{v}. \tag{2.10}$$

Then the equation for $\phi(\mathbf{k}, t)$ reads as follows (see ref. 3 for details)

$$\frac{\partial \phi}{\partial t} = B(\rho)\sqrt{\theta(t)}\, \mathfrak{I}(\phi, \phi) + \Lambda_{FP}\phi \tag{2.11}$$

where

$$\mathfrak{I}(\phi, \phi) = \int_{S^2} \frac{d\mathbf{n}}{4\pi} \left[\phi(\mathbf{k}_-)\, \phi(\mathbf{k}_+) - \phi(0)\, \phi(\mathbf{k})\right],$$

$$\mathbf{k} = \frac{1+e}{4}(\mathbf{k} - |\mathbf{k}|\, \mathbf{n}), \qquad \mathbf{k}_+ = \mathbf{k} - \mathbf{k}_- \tag{2.12}$$

Equalities (2.1) lead to the following formulas:

$$\rho = \phi(0, t), \qquad \rho\mathbf{u} = [i\nabla_{\mathbf{k}}\phi(\mathbf{k}, t)]_{\mathbf{k}=0}, \qquad 3\rho\theta = -\rho\, |\mathbf{u}|^2 - [\Delta_{\mathbf{k}}\phi(\mathbf{k}, t)]_{\mathbf{k}=0}, \tag{2.13}$$

whereas the operators defined in (2.9) read

$$\Lambda_{FP}^{(1)}\phi = -F\, |\mathbf{k}|^2\, \phi, \qquad \Lambda_{FP}^{(2)}\phi = -\frac{1}{\tau}(\theta_b\, |\mathbf{k}|^2\, \phi + \mathbf{k}\cdot\nabla_{\mathbf{k}}\phi), \tag{2.14}$$

3. MOMENTS OF THE COLLISION INTEGRAL

Our aim in this section is to find a connection between the moments $m_{i_1 i_2 \cdots i_n}$ of $f(\mathbf{v})$ and the corresponding moments $M_{i_1 i_2 \cdots i_n}$ of $Q(f, f)$, where

$$m_{i_1 i_2 \cdots i_n} = \int_{\mathbb{R}^3} f(\mathbf{v})\, v_{i_1} v_{i_2} \cdots v_{i_n}\, d\mathbf{v}, \qquad M_{i_1 i_2 \cdots i_n} = \int_{\mathbb{R}^3} Q(f, f)\, v_{i_1} v_{i_2} \cdots v_{i_n}\, d\mathbf{v}. \tag{3.1}$$

Let $f(\mathbf{v})$ be a given function, $g(\mathbf{v}) = Q(f, f)$. The two-parameter transformation

$$f(\mathbf{v}) = A\tilde{f}(\mathbf{v}-\mathbf{a}), \qquad A \in \mathbb{R}_+, \qquad \mathbf{a} \in \mathbb{R}^3, \tag{3.2}$$

obviously leads to

$$g(\mathbf{v}) = A^2\tilde{g}(\mathbf{v}-\mathbf{a}), \qquad \tilde{g} = Q(\tilde{f}, \tilde{f}). \tag{3.3}$$

If $A = \rho$, $\mathbf{a} = \mathbf{u}$ in (3.2) then

$$\int_{\mathbb{R}^3} \tilde{f}(\mathbf{v})\, d\mathbf{v} = 1, \qquad \int_{\mathbb{R}^3} \tilde{f}(\mathbf{v})\, \mathbf{v}\, d\mathbf{v} = 0.$$

Hence it suffices to consider the case

$$\rho = \int_{\mathbb{R}^3} f(\mathbf{v})\, d\mathbf{v} = 1, \qquad \rho\mathbf{u} = \int_{\mathbb{R}^3} f(\mathbf{v})\, \mathbf{v}\, d\mathbf{v} = 0. \tag{3.4}$$

without any loss of generality. Then

$$\phi(\mathbf{k}, t) = \int_{\mathbb{R}^3} f(\mathbf{v}, t)\, e^{-i\mathbf{k}\cdot\mathbf{v}}\, d\mathbf{v} = 1 + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} m_{i_1 i_2 \cdots i_n} k_{i_1} k_{i_2} \cdots k_{i_n}. \tag{3.5}$$

Here and below the usual rule of summation from 1 to 3 over repeated indices is assumed.

Remark. In this section we are not interested in the convergence of the formal expansion (3.5); the series is simply a convenient tool to evaluate exactly the moments of the collision integral. The same formulas (see below) are valid in the case when the function $f(\mathbf{v})$ has only a finite number of moments and thus the series (3.5) has no meaning at all. Our approach is merely a modification of methods developed earlier for the elastic case.[10]

Similarly we obtain:

$$\Psi(\mathbf{k}, t) = \mathfrak{I}(\phi, \phi) = \int_{\mathbb{R}^3} Q(f, f)\, e^{-i\mathbf{k}\cdot\mathbf{v}}\, d\mathbf{v} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} M_{i_1 i_2 \cdots i_n} k_{i_1} k_{i_2} \cdots k_{i_n}. \tag{3.6}$$

since $\Psi(0) = 0$, $[\nabla_{\mathbf{k}} \Psi(\mathbf{k})]_{\mathbf{k}=0} = 0$. We rewrite (3.5) as

$$\phi(\mathbf{k}) = 1 + \tilde{\phi}(\mathbf{k}), \qquad \tilde{\phi}(\mathbf{k}) = O(|\mathbf{k}|^2). \tag{3.7}$$

Then

$$\Psi(\mathbf{k}) = -L\phi + O(|\mathbf{k}|^4), \tag{3.8}$$

where

$$L\phi = \int_{S^2} \frac{d\mathbf{n}}{4\pi} [\phi(\mathbf{k}) - \phi(\mathbf{k}_+) - \phi(\mathbf{k}_-)]. \tag{3.9}$$

The linear operator L transforms any homogeneous polynomial $m_{i_1 i_2 \cdots i_n} k_{i_1} k_{i_2} \cdots k_{i_n}$ into a homogeneous polynomial of the same order. Therefore we obtain very simple equalities for moments of order $n = 2, 3$:

$$M_{ij} k_i k_j = -m_{ij} L k_i k_j, \qquad M_{ijl} k_i k_j k_l = -m_{ijl} L k_i k_j k_l \tag{3.10}$$

The operator L is isotropic. Therefore its (tensor) eigenfunctions are tensor spherical harmonics and we obtain:

$$\begin{aligned} L\,|\mathbf{k}|^2 &= \lambda_1\,|\mathbf{k}|^2, \\ L(k_i k_j - \tfrac{1}{3}\,|\mathbf{k}|^2\,\delta_{ik}) &= \lambda_2 (k_i k_j - \tfrac{1}{3}\,|\mathbf{k}|^2\,\delta_{ik}), \\ L\,|\mathbf{k}|^2\,\mathbf{k} &= \lambda_3\,|\mathbf{k}|^2\,\mathbf{k}, \\ L q_{ijl} &= \lambda_4 q_{ijl} \end{aligned} \tag{3.11}$$

where

$$q_{ijl} = k_i k_j k_l - \frac{|\mathbf{k}|^2}{5} (k_i \delta_{jl} + k_j \delta_{il} + k_l \delta_{ij}), \tag{3.12}$$

and λ_i $(i = 1, 2, 3, 4)$ are the corresponding eigenvalues. All eigenvalues were already found in ref. 3; one can however easily compute λ_i $(i = 1, 2, 3, 4)$ by using just the above equalities. Multiplying (scalarly in the case of vectors) the equalities in (3.11) by 1, $k_i k_j$, $\mathbf{k}$, $k_i k_j k_l$, respectively, we obtain:

$$\begin{aligned} \lambda_1 &= \frac{1}{|\mathbf{k}|^2} L\,|\mathbf{k}|^2, \\ \lambda_2 &= \frac{3}{2} \frac{k_i k_j}{|\mathbf{k}|^4} L \left(k_i k_j - \frac{1}{3}\,|\mathbf{k}|^2\,\delta_{ik} \right), \\ \lambda_3 &= \frac{\mathbf{k}}{|\mathbf{k}|^4} \cdot L\,|\mathbf{k}|^2\,\mathbf{k}, \\ \lambda_4 &= \frac{5}{2} \frac{k_i k_j k_l}{|\mathbf{k}|^6} L q_{ijl} \end{aligned} \tag{3.13}$$

where L is given by Eq. (3.9). Hence we need to evaluate four scalar integrals (3.13) only. This can be easily done by using the following equalities:

$$|\mathbf{k}_-|^2 = s(1-\epsilon)^2\,|\mathbf{k}|^2, \qquad |\mathbf{k}_+|^2 = [1-s(1-\epsilon^2)]\,|\mathbf{k}|^2,$$
$$\mathbf{k}_-\cdot\mathbf{k} = s(1-\epsilon)\,|\mathbf{k}|^2, \qquad \mathbf{k}_+\cdot\mathbf{k} = [1-s(1-\epsilon)]\,|\mathbf{k}|^2, \tag{3.14}$$

where $\epsilon = (1-e)/2$ and $0 \leqslant s \leqslant 1$ is given by:

$$s = \frac{1}{2}\left(1-\frac{\mathbf{k}\cdot\mathbf{n}}{|\mathbf{k}|}\right)$$

By using the identity:

$$\int_{S^2} \frac{d\mathbf{n}}{4\pi}\, F\left(\frac{1}{2}\left(1-\frac{\mathbf{k}\cdot\mathbf{n}}{|\mathbf{k}|}\right)\right) = \int_0^1 F(s)\,ds.$$

we can easily evaluate the integrals (3.13) and obtain:

$$\lambda_1 = \epsilon(1-\epsilon), \qquad \lambda_2 = \frac{1}{2}(1-\epsilon^2), \qquad \lambda_3 = \frac{(2+7\epsilon)(1-\epsilon)}{3}, \qquad \lambda_4 = \frac{3}{4}(1-\epsilon^2). \tag{3.15}$$

In order to formulate the final result of this section, we denote

$$g(\mathbf{v}) = Q(f, f), \qquad (h_1, h_2) = \int_{\mathbb{R}^3} h_1(\mathbf{v})\,h_2(\mathbf{v})\,d\mathbf{v}. \tag{3.16}$$

where $Q(f, f)$ is given by Eq. (2.2). The lower moments of $f(\mathbf{v})$ and $g(\mathbf{v})$ are denoted by:

$$\rho = (f, 1), \quad \rho\mathbf{u} = (f, \mathbf{v}), \quad p = \rho\theta = \frac{1}{3}(f, |\mathbf{c}|^2), \quad P = \frac{1}{3}(g, |\mathbf{c}|^2), \quad \mathbf{c} = \mathbf{v}-\mathbf{u},$$

$$q_{ij} = \left(f, c_i c_j - \frac{|\mathbf{c}|^2}{3}\delta_{ij}\right), \qquad Q_{ij} = \left(g, c_i c_j - \frac{|\mathbf{c}|^2}{3}\delta_{ij}\right),$$

$$q_i = (f, c_i\,|\mathbf{c}|^2), \qquad Q_i = (f, c_i\,|\mathbf{c}|^2);$$

$$q_{ijk} = (f, \Gamma_{ijk}), \quad Q_{ijk} = (g, \Gamma_{ijk}), \quad \Gamma_{ijl} = c_i c_j c_l - \frac{|\mathbf{c}|^2}{5}(c_i\delta_{jl} + c_j\delta_{il} + c_l\delta_{ij}).$$

Theorem 3.1. All moments of the collision integral $Q(f, f)$ of order $n \leqslant 3$ are uniquely determined by the moments of $f(\mathbf{v})$ through the following relations:

$$(g, 1) = 0, (g, \mathbf{v}) = 0, \quad P = -\epsilon(1-\epsilon)\, p, \quad \mathbf{c} = \mathbf{v} - \mathbf{u},$$

$$Q_{ij} = -\frac{1}{2}(1-\epsilon^2)\, \rho q_{ij}, \quad Q_i = -\frac{(2+7\epsilon)(1-\epsilon)}{3}\, \rho q_i, \quad Q_{ijk} = -\frac{3}{4}(1-\epsilon^2)\, \rho q_{ijk}. \tag{3.17}$$

provided $(|f|, 1+|\mathbf{v}|^4) < \infty$.

Proof. The proof follows from the above considerations since

$$\int_{\mathbb{R}^3} f(\mathbf{v})\, h(\mathbf{v}-\mathbf{u})\, d\mathbf{v} = \rho \int_{\mathbb{R}^3} \tilde{f}(\mathbf{v})\, h(\mathbf{v})\, d\mathbf{v},$$

where $\tilde{f}(\mathbf{v})$ in (3.2) corresponds to $A = \rho$, $\mathbf{a} = \mathbf{u}$. The equality (3.8) holds if we assume that $(|f|, 1+|\mathbf{v}|^4) < \infty$. All constant factors in (3.16) are equal to the corresponding eigenvalues (3.15). Thus theorem 1 is proved.

Thus all lower moments ($n \leqslant 3$) of $Q(f, f)$ are computed. Similar computations can be made for moments of order $n > 3$, but the resulting formulas are more complicated. On the other hand, Eq. (2.1) admits a wide class of isotropic solutions $f = f(|\mathbf{v}|, t)$; moreover, its steady state solutions are obviously isotropic in velocity space. Then the computation of moments becomes much easier, as we shall see in the next section.

4. ISOTROPIC SOLUTIONS

If $f = f(|\mathbf{v}|, t)$ in (2.1) then the characteristic function (2.11) can be written as

$$\phi = \phi(x, t), \qquad x = \frac{|\mathbf{k}|^2}{2}. \tag{4.1}$$

Equalities (2.12)–(2.14) lead to

$$\mathfrak{I}(\phi, \phi) = \int_0^1 ds[\phi((1-\epsilon)^2 sx)\, \phi([1-(1-\epsilon^2)\, s]\, x) - \phi(0)\, \phi(x)],$$

$$\rho = \phi(0, t), \qquad \theta = -\phi_x(0, t) \tag{4.2}$$

$$\Lambda_{FP}^{(1)} \phi = -2Fx\phi, \qquad \Lambda_{FP}^{(2)} \phi = -\frac{2}{\tau}\, x \left(\theta_b \phi + \frac{\partial \phi}{\partial x} \right),$$

where $\epsilon=(1-e)/2$. Without any loss of generality we assume that

$$\rho=(f,1)=\phi(0,t)=1. \tag{4.3}$$

Then

$$\phi\left(\frac{|\mathbf{k}|^2}{2},t\right)=\int_{\mathbb{R}^3} f(|\mathbf{v}|,t)\,e^{-i\mathbf{k}\cdot\mathbf{v}}\,d\mathbf{v}=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n!)}\,m_{2n}\,|\mathbf{k}|^{2n}, \tag{4.4}$$

or, equivalently,

$$\begin{gathered}\phi(x,t)=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\,\phi_n x^n,\\ \phi_n=\frac{2^n n!}{(2n)!}\,m_{2n}=\frac{m_{2n}}{(2n-1)!!},\\ m_{2n}=\int_{\mathbb{R}^3} f(|\mathbf{v}|,t)\,|\mathbf{v}|^{2n}\,d\mathbf{v}.\end{gathered} \tag{4.5}$$

On the other hand,

$$\mathfrak{I}(\phi,\phi)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\,u_n x^n,\qquad u_n=\frac{M_{2n}}{(2n-1)!!},\qquad M_{2n}=\int_{\mathbb{R}^3} Q(f,f)\,|\mathbf{v}|^{2n}\,d\mathbf{v}. \tag{4.6}$$

Hence, to compute M_{2n} (the isotropic moments of the collision integral) it suffices to express u_n through ϕ_n $(n=1,2,...)$. Substituting (26) into the integral (23) one can easily obtain (see also, Section 5 of ref. 3):

$$u_n=-\lambda_n\phi_n+\sum_{k=1}^{n-1} H(k,n-k)\,\phi_k\phi_{n-k}, \tag{4.7}$$

where

$$\lambda_n=\int_0^1\,\{1-(1-\epsilon)^{2n}\,s^n-[1-(1-\epsilon^2)\,s]^n\} \tag{4.8}$$

and

$$\begin{gathered}H(k,n-k)=\binom{n}{k}(1-\epsilon)^{2(n-k)}\int_0^1 ds\,s^{n-k}[1-(1-\epsilon)^2 s]^k\\ n=1,2,...\quad 1\leqslant k\leqslant n-1.\end{gathered} \tag{4.9}$$

The evaluation of the integrals yields

$$\lambda_n = 1 - \frac{1}{n+1} \left[(1-\epsilon)^{2n} + \frac{1-\epsilon^{2(n+1)}}{1-\epsilon^2} \right],$$

$$H(k, n-k) = \frac{(1-\epsilon)^{2(n-k)}}{n+1} \sum_{l=0}^{k} \binom{n-k+l}{l} \epsilon^{2l}, \quad n = 1, 2,\dots \quad 1 \leqslant k \leqslant n-1. \tag{4.10}$$

The result can be formulated as

Theorem 4.1. If $f(|\mathbf{v}|) \geqslant 0$ possesses all moments m_{2n} (4.5), $n = 0, 1, 2,\dots$, and $m_0 = 1$, then the corresponding moments M_{2n} (4.6) of the collision integral are given by the equalities (4.5)–(4.10).

Proof. The proof was already given above. We need only to mention that the convergence of the Taylor series (4.5), (4.6) plays no role in the derivation of (4.7)–(4.10). The formulas hold even in the case of a function $f(|\mathbf{v}|)$ with a finite number of moments.

The Fokker–Planck terms (4.2) lead to the following equalities

$$\begin{aligned} \Lambda_{FP}^{(1)} \phi &= 2F \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} [n\phi_{n-1}]\, x^n, \\ \Lambda_{FP}^{(2)} \phi &= -\frac{2}{\tau} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} [n(\theta_b \phi_{n-1} - \phi_n)]\, x^n, \end{aligned} \tag{4.11}$$

where $\phi(x, t)$ is given by the series (4.5). Let us consider now the initial value problem in velocity space

$$\frac{\partial f}{\partial t} = B(\rho) \sqrt{\theta(t)}\, Q(f, f) + L_{FP} f, \qquad f_{|t=0} = f_0(\mathbf{v}) \tag{4.12}$$

The substitution

$$f(\mathbf{v}, t) = \rho \tilde{f}(\mathbf{v}, \tilde{t}), \quad \tilde{t} = \rho B(\rho)\, t, \quad \rho = (f_0, 1) \tag{4.13}$$

yields

$$\frac{\partial \tilde{f}}{\partial \tilde{t}} = \sqrt{\theta(\tilde{t})}\, Q(\tilde{f}, \tilde{f}) + \tilde{L}_{FP} \tilde{f}, \qquad \tilde{f}_{|t=0} = \tilde{f}_0(\mathbf{v}), \tag{4.14}$$

where

$$\tilde{L}_{FP} = \frac{1}{B\rho} L_{FP}, \qquad (\tilde{f}, 1) = (\tilde{f}_0, 1) = 1 \tag{4.15}$$

Omitting tildes we note that the problem (4.12) always reduces to the case $\rho = 1$, $B(\rho) = 1$ with the corresponding scaling of (4.15) of the Fokker–Planck operator. Therefore we consider below just the case $\rho = 1$, $B(\rho) = 1$.

If $f_0 = f_0(|\mathbf{v}|)$ then the Fourier representation of (4.12) reads as

$$\frac{\partial \phi(x, t)}{\partial t} = \sqrt{\theta(t)}\, \Im(\phi, \phi) + \Lambda_{FP}\phi, \qquad \phi_{|t=0} = \phi_0(x) \tag{4.16}$$

where the notations (4.1), (4.2) are used. The solution of this problem can be constructed in the form of a power series

$$\phi(x, t) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \phi_n(t)\, x^n, \qquad \phi_0 = 1, \quad \phi_1 = \theta(t), \tag{4.17}$$

Provided the series:

$$\phi_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \phi_n^{(0)}(t)\, x^n, \qquad \phi_0^{(0)} = 1 \tag{4.18}$$

has a non-zero radius of convergence. If $\Lambda_{FP} = \Lambda_{FP}^{(1)}$ (4.2) then we obtain:

$$\begin{gathered} \frac{d\theta}{dt} + \lambda_1 \theta^{3/2} = 2F, \qquad \phi_1 = \theta, \\ \frac{d\phi_n}{dt} + \lambda_n \theta^{1/2} \phi_n = \theta^{1/2} \sum_{k=1}^{n-1} H(k, n-k)\, \phi_k \phi_{n-k} + 2Fn\phi_{n-1}, \quad n = 2,... \end{gathered} \tag{4.19}$$

where λ_n and $H(k, n-k)$ are given by Eqs. (4.8) and (4.9). If $\Lambda_{FP} = \Lambda_{FP}^{(2)}$ (4.2) the corresponding equations read as follows:

$$\begin{gathered} \frac{d\theta}{dt} + \left(\lambda_1 \theta^{1/2} + \frac{2}{\tau}\right)\theta = \frac{2}{\tau}\theta^b, \qquad \phi_1 = \theta, \\ \frac{d\phi_n}{dt} + \left(\lambda_n \theta^{1/2} + \frac{2n}{\tau}\right)\phi_n = \theta^{1/2} \sum_{k=1}^{n-1} H(k, n-k)\, \phi_k \phi_{n-k} + \frac{2n}{\tau}\theta_b \phi_{n-1}, \quad n = 2,... \end{gathered} \tag{4.20}$$

with the same notation. Equations (4.19) and (4.20) with initial conditions $\phi_n = \phi_n^{(0)}$, $n = 1,...$, given by the series (4.18) can be obviously solved in a recurrent fashion, beginning with $\phi_1 = \theta(t)$. The equation for $\theta(t)$ reads in both cases (4.19) and (4.20) as follows:

$$\frac{d\theta}{dt} = A(\theta_*) - A(\theta), \quad A'(\theta) > 0 \qquad \text{if} \quad \theta > 0, \tag{4.21}$$

where θ_* is a steady state temperature. Hence $\theta(t)$ is a monotone function such that $\theta(0) = \theta_0 = \phi_1^{(0)}$, $\theta(t) \to \theta_*$ as $t \to \infty$, since $\theta = \theta_*$ is a stable stationary solution. The equations for $\phi_n(t)$, $n \geqslant 2$, can be written as

$$\frac{d\phi_n}{dt} + a_n(t)\, \phi_n = b_n(t), \qquad n = 2,..., \quad a_n > 0, \quad b_n > 0 \tag{4.22}$$

where a_n and b_n are known if we solve the equations recursively. Therefore

$$\begin{aligned} \phi_n(t) &= \phi_n(0)\, e^{-A_n(t)} + \int_0^t d\tau\, b_n(\tau)\, e^{-[A_n(t) - A_n(\tau)]}, \qquad n = 2,..., \\ A_n &= \int_0^t ds\, a_n(s). \end{aligned} \tag{4.23}$$

By usual induction arguments we can easily prove the stability property

$$\phi_n(t) \to \phi_n^*, \qquad t \to \infty. \tag{4.24}$$

where $\phi_n^* = b_n(\infty)/a_n(\infty)$. Similar equations for tensor moments of order $n \leqslant 3$ follow from Theorem 3.1. It is clear, however, that all "anisotropic" moments of the solutions of (2.1) tend to zero as $t \to \infty$. Therefore, in order to study the asymptotic steady state solution, we may restrict ourselves to the isotropic case (4.16). Then the steady state solution $\phi(x)$ is given by the equation

$$\sqrt{\theta}\, \mathfrak{I}(\phi, \phi) + \Lambda_{FP}\phi = 0, \qquad \phi(0) = 1, \quad \theta = -\phi'(0) \tag{4.25}$$

The solution $\phi(x)$ reads as

$$\phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, \phi_n x^n, \tag{4.26}$$

where $\phi_0 = 1$, and $\phi_1 = \theta$ is given, in the two cases, by the equalities

$$\lambda_1 \theta^{3/2} = 2F, \qquad \text{or} \quad \left(\lambda_1 \theta^{1/2} + \frac{2}{\tau}\right)\theta = \frac{2}{\tau}\,\theta_b, \quad \lambda_1 = \epsilon(1-\epsilon), \tag{4.27}$$

whereas ϕ_n, $n \geqslant 2$, are given by the recurrence relations ($n = 2, 3,\ldots$)

$$\phi_n = \frac{1}{\lambda_n} \sum_{k=1}^{n-1} H(k, n-k)\, \phi_k \phi_{n-k} + 2\, \frac{Fn}{\lambda_n \theta^{1/2}}\, \phi_{n-1}, \quad n = 2,\ldots \tag{4.28}$$

or, respectively,

$$\phi_n = \frac{1}{\lambda_n \theta^{1/2} + 2n/\tau} \left[\theta^{1/2} \sum_{k=1}^{n-1} H(k, n-k)\, \phi_k \phi_{n-k} + \frac{2n}{\tau}\, \theta^b \phi_{n-1} \right]. \tag{4.29}$$

Now we must prove that the series (4.26) converges for $|x| < R$, $R > 0$. To this end we substitute (4.27) into (4.28), (4.29) and obtain

$$\phi_n = \frac{1}{\lambda_n} \sum_{k=1}^{n-1} H(k, n-k)\, \phi_k \phi_{n-k} + 2\, \frac{\lambda_1 n \theta^{1/2}}{\lambda_n}\, \phi_{n-1}, \tag{4.30}$$

$$\phi_n = \frac{\theta^{1/2}}{\lambda_n \theta^{1/2} + 2n/\tau} \sum_{k=1}^{n-1} H(k, n-k)\, \phi_k \phi_{n-k} + \frac{\lambda_1 \theta^{1/2} + \frac{2}{\tau}}{\lambda_n \theta^{1/2} + 2n/\tau}\, n\theta \phi_{n-1}. \tag{4.31}$$

where $n \geqslant 2$, and $\phi_1 = \theta$ is given by (4.27). Estimates on ϕ_n are based on the inequalities

$$\sum_{k=1}^{n-1} H(k, n-k) \leqslant \lambda_n, \qquad \frac{d}{dn} \lambda_n > 0, \quad 0 < \lambda_n < 1. \tag{4.32}$$

The first inequality was proved in ref. 3, the second and third inequalities follow from (4.8). First we consider $\{\phi_n\}$ defined by formulas (4.31) and prove that

$$0 < \phi_n \leqslant A^{n-1} \theta^n, \qquad A = 1 + \frac{\theta_b}{\theta}, \quad n = 1, 2,\ldots \tag{4.33}$$

The estimate obviously holds for $n = 1$. Then we use the induction argument and obtain

$$0 < \phi_n \leqslant A^{n-2} \theta^n \left[\frac{\theta^{1/2} \sum_{k=1}^{n-1} H(k, n-k)}{\lambda_n \theta^{1/2} + 2n/\tau} + \frac{1 + \frac{1}{2} \lambda_1 \theta^{1/2} \tau}{1 + \lambda_n \theta^{1/2} (\tau/2n)} \right] \tag{4.34}$$

therefore

$$0 < \phi_n \leqslant (A^{n-1} \theta^n)\, \frac{1}{A} \left[1 + 1 + \frac{1}{2}\, \lambda_1 \theta^{1/2} \tau \right]. \tag{4.35}$$

On the other hand, the second equality (4.27) is equivalent to $1+\frac{1}{2}\lambda_1\theta^{1/2}\tau = \theta_b/\theta$. Thus $\phi_n < A^{n-1}\theta^n$, for $n \geqslant 2$ and the estimate (4.33) is proved. Hence the series (4.26) with coefficients defined by (4.27) (second equality), (4.31) converges for all real (and complex) x.

Let us consider now the case (4.30) which formally corresponds to the limiting case $\tau \to \infty$ of (4.31). Assuming that

$$0 < \phi_n \leqslant n!\, B^{n-1}\theta^n, \qquad B = 1 + \frac{\lambda_1}{\lambda_2}, \qquad n = 1, 2, \ldots \tag{4.36}$$

we obtain by induction:

$$0 < \phi_n \leqslant n!\, B^{n-2}\theta^n \frac{1}{\lambda_n}\left[\sum_{k=1}^{n-1}\binom{n}{k}^{-1} H(k, n-k) + \lambda_1\right], \qquad n = 1, 2, \ldots, \tag{4.37}$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \geqslant n, \qquad 1 \leqslant k \leqslant n-1, \quad n = 2, \ldots. \tag{4.38}$$

Hence

$$\phi_n \leqslant n!\, B^{n-2}\theta^n \left(\frac{1}{n} + \frac{\lambda_1}{\lambda_n}\right) < n!\, B^{n-1}\theta^n, \qquad n = 2, \ldots, \tag{4.39}$$

because of inequalities (4.32) and the definition of B, (4.36). Thus the series (4.26) with coefficients defined by (4.27) (first equality), (4.30) has a non-zero radius of convergence $R \geqslant (B\theta)^{-1}$. Therefore we have found in each of the two cases $\Lambda_{FP}^{(i)}$ (4.2), $i = 1, 2$, a unique solution of Eq. (4.2), analytic at $x = 0$. If $\Lambda_{FP} = \Lambda_{FP}^{(1)}$ (4.2), then the function $\phi(x)$ for all $x > 0$ can be obtained by analytic continuation, since the series (4.26) has a finite radius of convergence and alternating signs. The result is formulated as

Theorem 4.2. The stationary kinetic equation in the Fourier representation

$$\sqrt{\theta}\, B(\rho)\, \Im(\phi, \phi) + \Lambda_{FP} = 0 \tag{4.40}$$

has for any given density $\rho = \phi(0)$ a unique isotropic solution $\phi(|\mathbf{k}|^2/2)$ analytic at $\mathbf{k} = 0$. The function $\phi(x)$ is defined (after reduction to the case $\rho = 1$) by Eqs. (4.26)–(4.31) for both cases $\Lambda_{FP} = \Lambda_{FP}^{(1,2)}$ (4.2).

To show that this solution $\phi(|\mathbf{k}|^2/2)$ corresponds to a "true" positive solution $f(|\mathbf{v}|)$ of Eq. (2.1), we need to prove that $\phi(|\mathbf{k}|^2/2)$ is really a characteristic function (Fourier transform of a probability density). We consider this question in the next section, where the functions $\phi(|\mathbf{k}|^2/2)$ and $f(|\mathbf{v}|)$ are studied in more detail.

5. MORE ABOUT STATIONARY SOLUTIONS

We restrict our considerations to the case $\Lambda_{FP} = \Lambda_{FP}^{(1)}$ (4.2). Then the stationary kinetic equation for f reads as follows

$$\sqrt{\theta}\, B(\rho)\, Q(f, f) + F \Delta f = 0 \tag{5.1}$$

One can choose arbitrarily two parameters $\rho > 0$ and $\mathbf{u} \in \mathbb{R}^3$ such that

$$\rho = \int_{\mathbb{R}^3} f(\mathbf{v}, t)\, d\mathbf{v}, \qquad \rho \mathbf{u} = \int_{\mathbb{R}^3} \mathbf{v} f(\mathbf{v}, t)\, d\mathbf{v}. \tag{5.2}$$

Then we denote

$$f(\mathbf{v}) = \rho \tilde{f}(\mathbf{v} - \mathbf{u}), \qquad F = \tilde{F} \rho B(\rho) \tag{5.3}$$

and reduce the equation to

$$\sqrt{\theta}\, Q(\tilde{f}, \tilde{f}) + \tilde{F} \Delta \tilde{f} = 0 \tag{5.4}$$

with additional conditions

$$\int_{\mathbb{R}^3} \tilde{f}(\mathbf{v}, t)\, d\mathbf{v} = 1, \qquad \int_{\mathbb{R}^3} \mathbf{v} \tilde{f}(\mathbf{v}, t)\, d\mathbf{v} = 0. \tag{5.5}$$

The Fourier representation of (5.4)–(5.5) reads as

$$\begin{aligned} &\sqrt{\theta}\, \Im(\phi, \phi) - F|\mathbf{k}|^2 \phi = 0, \qquad \phi(\mathbf{k}) = (f, e^{-i\mathbf{k}\cdot\mathbf{v}}), \\ &\phi(0) = 1 \qquad \nabla_{\mathbf{k}} \phi_{|\mathbf{k}=0} = 0, \qquad \Delta_{\mathbf{k}} \phi_{|\mathbf{k}=0} = -3\theta. \end{aligned} \tag{5.6}$$

Here and below tildes are omitted. The isotropic solution $\phi = \phi(|\mathbf{k}|^2)$ of (5.6) was constructed in Section 4. We have to prove now that there exists $f(\mathbf{v}) \in C^2(\mathbb{R}^3)$ such that $\phi(|\mathbf{k}|^2/2) = (f, e^{-i\mathbf{k}\cdot\mathbf{v}})$.

First we note that $\theta = (2F\lambda_1)^{2/3}$, where $\lambda_1 = \epsilon(1-\epsilon)$ in accordance with (4.27). Therefore Eq. (5.6) can be written as

$$\phi(x) = \frac{1}{1+\lambda_1 \theta x} \int_0^1 ds\, \phi((1-\epsilon)^2\, sx)\, \phi(x-(1-\epsilon^2)\, sx), \tag{5.7}$$

where $\theta = -\phi'(0)$ is given and $\phi(0) = 1$. We denote for brevity

$$\lambda = \lambda_1, \quad u(x) = \phi(x/\theta), \quad R[\phi] = \int_0^1 ds\, \phi((1-\epsilon)^2\, sx)\, \phi(x-(1-\epsilon^2)\, sx) \tag{5.8}$$

Then Eq. (5.7), rewritten in the new notation reads:

$$u(x) = \frac{1}{1+\lambda x} R[u], \qquad u(0) = -u'(0) = 1 \tag{5.9}$$

An alternative way to construct a solution of (5.9) is to use the iteration scheme

$$u_{n+1}(x) = \frac{1}{1+\lambda x} R[u_n], \qquad n = 0, 1, \ldots \tag{5.10}$$

If $u_0(x) \geqslant 0$ for all $x \geqslant 0$ and $u_0(0) = -u_0'(0) = 1$, then the same properties hold for $u_n(x) > 0$, $n \geqslant 1$. Moreover, if $u_1(x) \geqslant u_0(x) \geqslant 0$ $(0 \leqslant u_1 \leqslant u_0)$, then $u_{n+1}(x) \geqslant u_n(x) \geqslant 0$ $(0 \leqslant u_{n+1} \leqslant u_n)$ for all $n \geqslant 1$. Taking $u_0(x) = 1$ and $u_0(x) = e^{-x}$, we obtain a monotonically decreasing sequence

$$u_0(x) = 1, \qquad u_1(x) = \frac{1}{1+\lambda x}, \ldots \tag{5.11}$$

and a monotonically increasing sequence

$$u_0(x) = e^{-x}, \qquad u_1(x) = \frac{e^{-x}}{1+\lambda x} \frac{e^{2\lambda x}-1}{2\lambda x}, \ldots \tag{5.12}$$

The second sequence converges pointwise to a function $0 \leqslant u(x) \leqslant 1$ since $0 \leqslant u_n(x) \leqslant 1$ for all $x \geqslant 0$ and $n = 0, 1, \ldots$. On the other hand, for any $n \geqslant 0$ the function $u_n(x)$ can be expressed as

$$u_n(|\mathbf{k}|^2) = (f_n, e^{-i\mathbf{k}\cdot\mathbf{v}}), \qquad (f_n, 1) = \tfrac{1}{3}(f_n, |\mathbf{v}|^2) = 1, \tag{5.13}$$

where $f_n = f_n(|\mathbf{v}|) \geqslant 0$. This can be easily proved by induction provided $u_1(x)$ is a characteristic function. Then we note the following: if $u_n(x)$ is a characteristic function, then so is $R(u_n)$ in Eq. (5.10), since the operator R is the Fourier transform of the gain term of the collision integral (2.2). Moreover the function before the operator R in Eq. (5.10) is also a characteristic function and so is $u_{n+1}(x)$ as a product of two characteristic functions. Then, from general properties of characteristic functions (see, e.g., ref. 5) we conclude that the convergence $u_n(x) \to u(x)$ is uniform on $[0, \infty)$ and there exists a unique $f = f(|\mathbf{v}|) \geqslant 0$ such that

$$u(|\mathbf{k}|^2) = (f, e^{-i\mathbf{k}\cdot\mathbf{v}}), \qquad (f, 1) = \tfrac{1}{3}(f, |\mathbf{v}|^2) = 1. \tag{5.14}$$

On the other hand, the following estimate follows from the sequence in (5.11):

$$0 \leqslant u(x) \leqslant u_2(x) = \frac{1}{1+\lambda x} R\left[\frac{1}{1+\lambda x}\right] \leqslant 1. \tag{5.15}$$

An elementary computation shows that

$$u_2(x) \cong \text{const.} x^{-3} \log x, \qquad x \to \infty. \tag{5.16}$$

Therefore

$$\int_{\mathbb{R}^3} (1+|\mathbf{k}|^2)\, u\left(\frac{|\mathbf{k}|^2}{2}, t\right) d\mathbf{k} < \infty \tag{5.17}$$

and

$$f(\mathbf{v}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} u\left(\frac{|\mathbf{k}|^2}{2}, t\right) e^{i\mathbf{k}\cdot\mathbf{v}}\, d\mathbf{k} \in C^2(\mathbb{R}^3). \tag{5.18}$$

Hence we have constructed a classical solution of (5.6). It is easy to show that the solution is unique in the class of functions $f(\mathbf{v})$ such that

$$f(\mathbf{v}) \geqslant 0, \qquad f(\mathbf{v}) \in C^2(\mathbb{R}^3), \qquad (f, 1+|\mathbf{v}|^2) < \infty, \tag{5.19}$$

provided the conditions in (5.5) are fulfilled. The result is formulated as

Theorem 5.1. Equation (5.6) has a unique solution satisfying (5.5), (5.19). The solution is uniquely defined by its power moments (4.5)

$$\phi_n = \frac{(f, |\mathbf{v}|^{2n})}{(2n-1)!!}, \qquad n = 1, 2, \ldots; \quad \phi_0 = 1, \tag{5.20}$$

given by the recurrence formulas (4.27), (4.28).

Similar results can be proved for the second case in (2.10), $L_{FP} f = L^{(2)}_{FP}$, which needs, however, more technicalities and is not considered here. We note only that in this case the solution $f(|\mathbf{v}|)$ has the usual Maxwellian tail (roughly speaking, $f \cong \exp(-\alpha\, |\mathbf{v}|^2)$ as $|\mathbf{v}| \to \infty$), as follows from the estimates for the moments, Eq. (4.33). This is not the case, however, for a $f(|\mathbf{v}|)$ satisfying Eq. (5.1) (as one can guess from the much weaker estimates (4.36) for the moments).

To study the asymptotics for large speeds ($|\mathbf{v}| \to \infty$) it is convenient to use the two-sided Laplace (instead of Fourier) transform. We denote

$$\psi\left(\frac{|\mathbf{k}|^2}{2}\right) = \int_{\mathbb{R}^3} f(|\mathbf{v}|)\, e^{\mathbf{k}\cdot\mathbf{v}}\, d\mathbf{v} = \frac{4\pi}{|\mathbf{k}|} \int_0^\infty f(r) \sinh(|\mathbf{k}|\, r)\, r\, dr. \tag{5.21}$$

and apply this transformation to Eq. (5.4). The result can be easily obtained from Eq. (5.6) by changing $\mathbf{k}$ to $i\,\mathbf{k}$, so that $\psi(x) = \phi(-x)$ satisfies the equation

$$\psi(x) = \frac{1}{1-\lambda\theta x} R[\psi], \qquad \psi(0) = \psi'(0) = \theta, \qquad x = |\mathbf{k}|^2/2 \geqslant 0. \tag{5.22}$$

We note that $\sinh(|\mathbf{k}|\, r) > |\mathbf{k}|\, r$ and therefore $\psi(x) \geqslant 1$ for all $x \geqslant 0$. Therefore

$$R[\psi] \geqslant 1, \quad \Rightarrow \psi(x) \geqslant \frac{1}{1-\lambda\theta x}. \tag{5.23}$$

Hence,

$$\int_0^\infty f(r)\, e^{|\mathbf{k}|\, r} r\, dr > \frac{|\mathbf{k}|}{\pi(2-\lambda_1\theta\, |\mathbf{k}|^2)}, \tag{5.24}$$

and, accordingly, the continuous function $f(|\mathbf{v}|) \exp[\sqrt{2/(\lambda_1\theta)}\, |\mathbf{v}|] \geqslant 0$ cannot be integrable. One can assume that

$$f(|\mathbf{v}|) \cong \text{const.}\ |\mathbf{v}|^p \exp\left[-\sqrt{\frac{2}{\lambda_1\theta}}\, |\mathbf{v}|\right], \qquad |\mathbf{v}| \to \infty \tag{5.25}$$

Then

$$\psi(x) \cong \text{const.}\ (1-\lambda_1\theta x)^{-(p+2)}, \qquad x \to (\lambda_1\theta)^{-1}. \tag{5.26}$$

The integral

$$R[\psi] = \frac{1}{x} \int_0^x dy\, \psi((1-\epsilon)^2 y)\, \psi(x-(1-\epsilon^2)\, y) \tag{5.27}$$

behaves for $x \to (\lambda_1 \theta)^{-1}$ as

$$\begin{aligned} R[\psi] &\cong (\lambda_1 \theta)\, \psi(0) \int_0^x dy\, \psi(x-(1-\epsilon^2)\, y) \\ &\cong \text{const.} \psi(0)\, \frac{1}{1-\epsilon^2}\, \frac{1}{p+1}\, (1-\lambda_1 \theta x)^{-(p+1)}, \end{aligned} \tag{5.28}$$

with the same constant as in (5.26). Noting that $\psi(0) = 1$, we obtain from (5.22) the equality

$$1 = \frac{1}{(1-\epsilon^2)(p+1)}. \tag{5.29}$$

Hence, $p = \epsilon^2 (1-\epsilon^2)^{-1}$ in (5.25), and we obtain, at a formal level, the leading asymptotic term of the distribution function $f(|\mathbf{v}|)$ for $|\mathbf{v}| \to \infty$ up to a constant factor. This specific asymptotics is lost if one uses a conventional expansion

$$f(|\mathbf{v}|) \cong (2\pi\theta)^{-3/2}\, e^{-\frac{|\mathbf{v}|^2}{\theta}} \left[1 + \epsilon^2 h\left(\frac{|\mathbf{v}|^2}{\theta} \right) \right] \tag{5.30}$$

for small ϵ.[4] Such an expansion can be easily constructed on the basis of the Fourier transformed equation (5.6). We denote

$$\phi(x) = u(\theta x) = e^{-\theta x} y(\theta x), \qquad y(0) = 1, \qquad y'(0) = 0,$$

and obtain the equation for $y(x)$:

$$y(x)(1+\lambda x) = \int_0^1 ds\, e^{2\lambda s x} y((1-\epsilon)^2\, sx)\, y(x-(1-\epsilon^2)\, sx), \tag{5.31}$$

where $\lambda = \epsilon(1-\epsilon)$. Assuming that

$$y(x) = 1 + w(x; \epsilon),$$

we obtain after obvious calculations:

$$w(x; \epsilon) = 2\epsilon^2 x^2 + O(\epsilon^3).$$

This term corresponds to the result of ref. 4. The terms of higher orders in ϵ can be easily found from Eq. (5.31). Then the inverse Fourier transform yields the expansion (5.30).

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